Actions of Pointed Hopf Algebras

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1 Introduction

Throughout this paper H is a finite dimentional Hopf algebra over a field k, and A is a associative k-algebra.

Definition 1.1 *It is said that H acts on A, if A is left H-module and for any $h \in H, \quad a, b \in A$*

$$h(ab) = \sum_h (h_{(1)}a)(h_{(2)}b), \quad h1 = \varepsilon(h),$$

where $\varepsilon : H \to k$ - counit and Δ - comultiplication:

$$\Delta(h) = \sum_h h_{(1)} \otimes h_{(2)} \in H \otimes H.$$

Often A is called H-module algebra.

Definition 1.2 *The invariants of H in A is the set A^H of those $a \in A$, that $ha = \varepsilon(h)a$ for each $h \in H$.*

Straightforvard computations shows, that A^H is the subalgebra of A. We refer reader to [5], [6] for the basic information concerning Hopf algebras and their actions on associative algebras.

As a generalization of the well-known fact for group actions the following question raised in [5] (Question 4.2.6.)

Question 1.3 *If A is a commutative k-algebra and H any finite dimensional Hopf algebra such that A is H-module algebra, is A integral over A^H ?*

If A is an affine algebra, then Artin-Tate lemma ensures that A^H is also affine.

Some positive answers to question 1.3 are known.

Theorem 1.4 ([2]) *Let H be a finite dimensional cocommutative Hopf algebra and let A be a commutative H-module algebra. Then A is integral extension of A^H.*

Some results on affine invariants were obtained without using integrality.

Definition 1.5 *Element $t \in H$ is called left integral, if $ht = \varepsilon(h)t$ for all $h \in H$.*

Note that $tA \subseteq A^H$.

Theorem 1.6 ([5], Theorem 4.3.7) *Let A be left Noetherian ring which is an affine k-algebra, and assume that A is an H-module algebra, such that $tA = A^H$. Then A^H is k-affine.*

As mentioned in [5], p. 48, if H is semisimple, then $tA = A^H$. By Maschke's theorem, H is semisimple if and only if $\varepsilon(t) \neq 0$ for some non-zero left integral t. Since the space $\int_H^l$ of left integrals in H is one-dimensional ([5], §2.2), the semisimplicity of H is equivalent to the fact that $\varepsilon(t) \neq 0$ for all non-zero left integrals.

The main result of the work is stated in the theorem 2.7. It gives positive answer to Question 1.3 in some partial cases. Let H be a pointed Hopf algebra and let A be an affine H-module algebra; if one of three conditions is satisfied, then A is integral over A^H:

1. H is commutative as an algebra,

2. char $k = p > 0$,

3. A is integral domain.

We recall that Hopf algebra H is called *pointed* if every simple subcoalgebra of H is one-dimentional; pointed Hopf algebra is called *connected* if it has only one simple subcoalgebra (one-dimensional). The examples of pointed Hopf algebras are given by group algebras, universal enveloping algebras. In fact, if G - group, then the only simple subcoalgebras of kG are those of the form kg, $g \in G$. At the same time universal enveloping algebras are examples of connected Hopf algebras: the only simple subcoalgebra of the universal enveloping algebra is $k1_H$.

Another important example of pointed Hopf algebras is represented by series of Hopf algebras $A_{N,\xi}$, where $N \geq 2$ – integer number and ξ – root

of unity of degree N, considered in [3] (see also section 3 of this paper). Note that with $N = 2, \xi = -1$, char $k \neq 2$ algebra $A_{2,-1}$ (sometimes called H_4) is the only Hopf algebra of minimal dimension neither cocommutative nor commutative ([5], example 1.5.6). In other words, algebra $A_{2,-1}$ is the minimal Hopf algebra not covered by theorem 1.4, but covered by theorem 2.7.

Notice that ideal generated by x in $A_{N,\xi}$ is nilpotent, therefore it lies in radical of $A_{N,\xi}$, i.e., $A_{N,\xi}$ is not semisimple. The same fact can be verified using argument of left integrals of H. Thus, theorem 2.7 is not covered by theorem 1.6.

In spite of numerous partial positive results it turned out that hypothesis 1.3 isn't true in general, e ven for pointed Hopf algebras. The counterexamples is built in section 3 for series of pointed Hopf algebras $A_{N,\xi}, N \geq 2$, mentioned at previous paragraphs [1] .

2 The main theorem

The proof of the theorem is based on properties of coradical filtration of an arbitrary coalgebra. We recall the basic facts.

Definition 2.1 *Coradical C_0 of coalgebra C is the (direct) sum of all simple subcoalgebras of C. Further by the induction for each $n \geq 1$ define*

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$$

Theorem 2.2 ([5], Theorem 5.2.2) *$\{C_n\}_{n\geq 0}$ is the family if subcoalgebras with the following properties:*

1. $C_{n-1} \subseteq C_n, \quad C = \bigcup\limits_{n\geq 0} C_n,$

2. $\Delta C_n \subseteq \sum\limits_{i=0}^{n} C_i \otimes C_{n-i}.$

Reader may find more detailed description of coradical filtration in [6], chapter 9.

[1] After the work was done authors became aware through e-mail by Susan Montgomery, that counterexample to hypothesis 1.3 was independently built by Zhu Shenglin. Also he obtained some positive results, solving problem 1.3. His paper "Integrality of module algebras over its invariants" should have been appeared at J.Algebra in 1996.

Let H be a pointed finite dimentional Hopf algebra over field k. Let $G = G(H)$ denote the set of *grouplike* elements of H, i.e.,

$$G = \{ \quad g \in H \setminus 0 \quad | \quad \Delta g = g \otimes g \quad \}.$$

It is known that elements of G are lineary independent, G is the group under multiplication arised from multiplication in H, and subalgebra generated by G is group Hopf algebra kG. Also kG is coradical of H.

By lemma 5.2.8 [5], coradical filtration $\{H_n\}$ of Hopf algebra H is Hopf filtration of Hopf algebra, i.e., $\Delta H_n \subseteq \sum_{i=0}^{n} H_i \otimes H_{n-i}, \quad H_n H_m \subseteq H_{n+m}$, $SH_n \subseteq H_n$ for all $n, m \geq 0$, if and only if H_0 is sub-Hopf algebra of H. If H – pointed finite dimensional Hopf algebra, then this condition is obviously satisfied. Moreover, coradical filtration is finite.

By theorem 5.4.1 from [5] (see also [7], [4] for reference), coradical filtration of pointed Hopf algebra H have additional properties. If $x \in H_m$, $m \geq 1$, then

$$x = \sum_{g,h \in G(H)} c_{g,h},$$

where

$$\Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w, \qquad (1)$$
$$w \in H_{m-1} \otimes H_{m-1}.$$

Note that if $a, b, g, h \in G$ and $c = ac_{g,h}b$, then by (1)

$$\Delta(c) = c \otimes agb + ahb \otimes c + w', \quad w' \in H_{n-1} \otimes H_{n-1}. \qquad (2)$$

Define $H^+ = \ker \varepsilon, \quad H_r^+ = H_r \cap H^+$. Let A^G denote subalgebra of G-invariants in A ($A^H \subseteq A^G$). Extension A/A^G is integral by Noether's theorem for group actions (H – finite dimensional, therefore G – finite group).

Before we start to prove main theorem we are going to obtain few auxiliary results.

Let I denote the ideal in H generated by elements of form $g - 1$ that $g \in G$.

Proposition 2.3 *I is Hopf ideal in H.*

Proof. If $g \in G$, then

$$\Delta(g-1) = g \otimes g - 1 \otimes 1 =$$
$$(g-1) \otimes g + 1 \otimes (g-1) \in I \otimes H + H \otimes I.$$

$S(I) \subseteq I$, because $S(g-1) = g^{-1} - 1$ and S is anti-homomorphism. This yields the proposition. $\square$

Proposition 2.4 *If J – Hopf ideal in H, then H/J – pointed Hopf algebra. Moreover, natural epimorphism of Hopf algebras $\pi : H \to H/J$ induces epimorphism of groups of grouplike elements $G(H) \to G(H/J)$.*

Proof. This statement is direct consequence of corollary 5.3.5 from [5]. □

Theorem 2.5 *Let one of three following conditions be satisfied:*

1. *char $k = p > 0$.*
2. *A is integral domain;*
3. *H – connected and commutative;*

Then there exists the chain of subalgebras $A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_n$ with following properties:

1. *each extension $A_i \supseteq A_{i+1}$ is integral;*
2. *if $x \in H_i^+$ then $x(A_i) = 0$.*

Proof. To construct this chain we start with defining $A_0 = A^G$. Both of necessary conditions are satisfied. Let the chain

$$A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_r, \quad r \geq 0,$$

be already constructed and let $x \in H_{r+1}^+$. By (1) we may assume that $x = c_{g,h}$, where $g, h \in G$. Then

$$\Delta(x) = x \otimes g + h \otimes x + \sum u_j \otimes v_j, \tag{3}$$

where $v_j, u_j \in H_r$. As $x \in H^+$, then by (3)

$$x = (1 \otimes \varepsilon)\Delta(x) = x + \sum u_j \varepsilon(v_j),$$

$$x = (\varepsilon \otimes 1)\Delta(x) = x + \sum \varepsilon(u_j) v_j.$$

Therefore,

$$\sum \varepsilon(u_j) v_j = \sum u_j \varepsilon(v_j) = 0,$$

and finally

$$\sum (u_j - \varepsilon(u_j)) \otimes (v_j - \varepsilon(v_j)) = \sum u_j \otimes v_j,$$

i.e., we may assume that $u_j, v_j \in H_r^+$.

If char $k=p>0$, then we define $A_{m+1}=A_m^p$. Really, by (3)

$$x(a^p)=h(a)^{p-1}x(a)+h(a)^{p-2}x(a)g(a)+\ldots$$
$$\ldots+x(a)g(a)^{p-1}=pa^{p-1}x(a)=0.$$

Suppose A is integral domain and char $k=0$. If $a\in A_r, b\in A$, then by (3)

$$x(ab)=x(a)b+ax(b),$$

i.e., $x:A_r\to A$ is derivation. By normalization lemma (see [1], chapter 5, §3, p.344), there exists subalgebra of polynomials $k[T_1,\ldots,T_d]$ in A_r such that extension $A_r/k[T_1,\ldots,T_d]$ is integral. In this case we have for each $f\in k[T_1,\ldots,T_d]$:

$$x(f)=\sum_{i=1}^{d}\frac{\partial f}{\partial T_i}a_i,\quad a_i\in A.$$

Therefore, for each integer $q\geq 1$

$$x^q(f)=\sum_{\substack{i_1+\ldots+i_d=q\\ i_s\geq 0}}\frac{\partial^q f}{\partial T_1^{i_1}\ldots\partial T_d^{i_d}}a_1^{i_1}\ldots a_d^{i_d}\quad +$$
$$+\quad\sum_{\substack{j_1+\ldots+j_d=l<q\\ j_s\geq 0}}\frac{\partial^l f}{\partial T_1^{j_1}\ldots T_d^{j_d}}\Psi_{j_1,\ldots,j_d},\tag{4}$$

where $\Psi_{j_1,\ldots,j_d}$ is the sum of monomials of form

$$x^{\alpha_{j_1}}(a_{j_1})\ldots x^{\alpha_{j_d}}(a_{j_d}),\quad \alpha_{j_1}+\ldots+\alpha_{j_d}=q-l.$$

H is finite dimensional, that is why there exists such integer m that

$$x^m=\sum_{i=1}^{m-1}\beta_i x^i,\quad \beta_i\in k.\tag{5}$$

Thus, by (4) and (5) for each $f\in k[T_1,\ldots,T_d]$ and each $i=1,\ldots,d$ we have:

$$\frac{\partial^m f}{\partial T_i^m}a_i^m+\sum_{1\leq l<m}\frac{\partial^l f}{\partial T_i^l}\Phi_l+\Lambda=0,\tag{6}$$

where Λ is the sum of all summands from (4) and (5) containing

$$\frac{\partial^q f}{\partial T_1^{j_1}\ldots\partial T_d^{j_d}}$$

as a multiplier, and besides one of the coefficients $j_s, s \neq i$, is not equal to zero. Substituting to (6) successivly $1, T_i, T_i^2, \ldots, T_i^m$, we get that

$$\Lambda = \Phi_1 = \ldots = \Phi_{m-1} = a_i^m = 0.$$

We use here the fact that char $k = 0$. As A – integral domain, so $a_i = 0$. Thus $x(k[T_1, \ldots, T_d]) = 0$ and we define $A_{r+1} = k[T_1, \ldots, T_d]$.

Suppose H – connected, commutative Hopf algebra and char $k = 0$, i.e., $g = h = 1$. By (3),

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum u_j \otimes v_j, \tag{7}$$

where $u_j, v_j \in H_r^+$.

We consider ideal HH_r^+ in H.

Lemma 2.6 *HH_r^+ is coideal in H.*

Proof. Let $u \in H, v \in H_r^+$. Since H_r – coalgebra, H_r^+ – it's coideal, i.e.,

$$\Delta(v) \in H_r \otimes H_r^+ + H_r^+ \otimes H_r,$$

and therefore,

$$\Delta(uv) = \Delta(u)\Delta(v) \in H \otimes (HH_r^+) + (HH_r^+) \otimes H. \quad \square$$

Assume that $x(A_r) \neq 0$. HH_r^+ acts as zero on A_r, so $x \notin HH_r^+$. Suppose $1, x, \ldots, x^{m-1}$ are linary independent modulo HH_r^+ and

$$x^m = \sum_{j=0}^{m-1} \alpha_j x^j + w, \quad w \in HH_r^+. \tag{8}$$

Choose k-basis $e_1, \ldots, e_d$ in H such that first elements $e_1, \ldots, e_{t-1}$ form k-basis for HH_r^+, $e_t = 1$, and $e_{t+1} = x, \ldots, e_{t+m-1} = x^{m-1}, d \geq t + m - 1$. Use Δ on (8). By (7), lemma 2.6 and commutativity of H,

$$(x \otimes 1 + 1 \otimes x)^m = \sum_{j=0}^{m-1} \alpha_j (x \otimes 1 + 1 \otimes x)^j + w', \tag{9}$$

$$w' \in H \otimes HH_r^+ + HH_r^+ \otimes H.$$

Note that commutativity of H is necessary only here to show, that w' really lies in $H\otimes HH_r^+ + HH_r^+\otimes H$. For each integer $q\geq 1$

$$(x\otimes 1+1\otimes x)^q=\sum_{i=0}^{q}\binom{q}{i}x^i\otimes x^{q-i}.$$

Subtracting from (9) equation

$$1\otimes x^m=\sum_{j=0}^{m-1}\alpha_j 1\otimes x^j+1\otimes w,$$

by (8), (9) we get

$$\sum_{i=1}^{m}\binom{m}{i}x^i\otimes x^{m-i}=\sum_{j=1}^{m-1}\alpha_j\sum_{i=1}^{j}\binom{j}{i}x^i\otimes x^{j-i}+w'', \tag{10}$$

$$w''\in H\otimes(HH_r^+)+(HH_r^+)\otimes H.$$

Since char $k=0, so\binom{m}{1}=m\neq 0$ in k. From this and (10) it follows that element $x\otimes x^{m-1}=e_{t+1}\otimes e_{t+m-1}$ in $H\otimes H$ is linear combination of elements $e_s\otimes e_{s'}$, where either s is less then $t+1$ or s' is less then $t+m-1$. But it is impossible, because elements $e_q\otimes e_{q'},\quad q,q'=1,\ldots d$, form basis of $H\otimes H$. This contradiction shows that $x(A_r)=0$. So in this case we define $A_{r+1}=A_r=\cdots=A^G$. $\square$

Notice that reasoning shown above is close to that used in [5], §5.5, §5.6.

We have come to main

Theorem 2.7 *Let A be an affine H-module algebra, H – finite dimensional pointed Hopf algebra and one of three conditions is satisfied:*

1. *char $k=p>0$;*
2. *H – commutative;*
3. *A – integral domain.*

Then extension A/A^H is integral.

Proof. Assume that H is commutative, then A^G is H-module algebra. It is sufficiently to show that A^G is stable under H-action. In fact, for each $x\in H,\quad a\in A$,

$$gx(a)=xg(a)=x(a),$$

i.e., $x(a) \in A^G$. Let I denote the ideal in H generated by the elements of form $g-1$ $(g \in G)$. By proposition 2.3 I is Hopf ideal. Obviously, it acts as zero on A^G. Hopf algebra H/I is pointed by proposition 2.4, moreover, it is connected. Thus the second case of this theorem is reduced to consideration of action of connected, commutative Hopf algebra H/I on algebra A^G. Now we apply theorem 2.5 to all cases. Let

$$A = A_{-1} \supseteq A_0 \supseteq \ldots \supseteq A_n$$

be constructed chain of subalgebras. By condition 1) of theorem 2.5, extension A/A_n is integral; by condition 2) $A_n \subseteq A^H$. $\square$

Note that if char $k = p > 0$, then $(A^G)^{p^{\dim H}} \subseteq A^H$. If H is commutative and char $k = 0$, then $A^H = A^G$.

3 Counterexample to hypothesis

Example 3.1 Hopf algebra H may be any from series $A_{N,\xi}$, $N \geq 2$. Algebra $A_{N,\xi}$ is generated by elements g, x with relations

$$g^N = 1, \quad x^N = 0, \quad xg = \xi gx, \tag{11}$$

where $\xi \in k$ – root of unity of degree N. Hopf algebra structure on $A_{N,\xi}$ is given as follows:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{N-1} = g^{-1},$$

$$\Delta(x) = g \otimes x + x \otimes 1, \quad \varepsilon(x) = 0, \quad S(x) = -g^{N-1}x.$$

We demand $\xi \neq 1$ and char $k = 0$. Algebra $A_{N,\xi}$ is pointed,

$$G(A_N) = \{1, g, g^2, \ldots, g^{N-1}\},$$

it is non-commutative and non-cocommutative.

Let A be commutative algebra generated by elements y, z with relation $z^2 = 0$. Define action $A_{N,\xi}$ on A:

$$g(y^n) = y^n, \quad g(y^n z) = \xi^{-1} y^n z, \quad x(y^n) = n y^{n-1} z, \quad x(y^n z) = 0.$$

Straightforvard computations shows the correctness of this action, i.e., $A_{N,\xi}(I) \subseteq I$, where I is ideal of free algebra $k < y, z >$ generated by elements $yz - zy, z^2$. We check that relations (11) in H are satisfied:

$$\xi g x(y^n) = \xi \xi^{-1} n y^{n-1} z = n y^{n-1} z = x g(y^n),$$

$$\xi g x(y^n z) = 0 = x g(y^n z),$$

$$g^N(y^n) = y^n, \quad g^N(y^n z) = \xi^{-N} y^n z = y^n z,$$

$$x^2(y^n) = n x(y^{n-1} z) = 0, \quad x^2(y^n z) = x(0) = 0,$$

i.e., $x^N(a) = x^2(a) = 0$ for any $a \in A$.

Obviously $A^G = k[y]$ and $A^H = k[y] \bigcap \ker x = k$. But extension A/A^H is not integral, because A is not finite k-module ($\dim_k A = \infty$).

4 Conclusion

As it was shown in example 3.1, hypothesis 1.3 is not true in general. Nevertheless all known examples of Hopf algebra action shows that if A – affine, then A^H is also affine algebra, although extension A/A^H is not always integral. So we ask

Question 4.1 *Finite dimensional Hopf algebra H acts on commutative affine algebra A. Is A^H affine?*

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